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SOME RESULTS ON SYMMETRIC BI-(σ,τ)-DERIVATIONS IN PRIME RINGS

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ABSTRACT

Let R be a 2-torsion free prime ring, U be a nonzero left ideal of R.Suppose there exists Symmetric bi-(σ,τ) derivations $F(.,.): R\times R\to R$, $G(.,.): R\times R\to R$, $H(.,.): R\times R\to R$ and $T(.,.): R\times R\to R$ such that f, g, h, t are traces of F, G, H, T respectively. (i) Then $g(x)h(y)=h(x)g(y)$, for all $x,y\in R$. If $g\neq 0$ then there exists $\mu\in C$ such that $h(x)=\mu g(x)$, for all $x\in R$. (ii) $f(x)h(y)=t(x)g(y)$, for all x, $y\in R$ moreover If $g\neq 0$ and $f\neq 0$ then there exists $\mu\in C$ such that $h(x)=\mu g(x)$ and $t(x)=\mu f(x)$, for all $x\in R$. (iii) If $g(x)=ah(x)+t(x)b$, for all$x\in R$. If $a\notin Z(R)$ and $b\notin Z(R)$ then there exists $\mu\in C$ such that $g(x)=[2\mu ab, a\sigma(x)+\tau(x)b]$, $h(x)=[\mu b,\sigma(x)]$ and $t(x)=[\mu(a),\tau(x)]$ for all $x\in R$. (iv) If $ag(x)+h(x)b=0$, for all $x\in R$. If $a\notin Z(R)$ and $b\notin Z(R)$ then there exists $\mu\epsilon C$ such that $g(x)=[\mu b,x]$ and $h(x)=[\mu a,x]$ for all$x\epsilon R$. Moreover if $g\neq 0$

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implies $ab\epsilon Z(R)$. (v) If g of non commutative R maps U in the center $Z(R)$, then $g = 0$.

Introduction: The concept of a symmetric bi-derivation has been introduced Maksa in [5]. Some recent results on properties of prime rings, semi-prime rings and near rings with derivations have been investigated in several ways [1, 4, 8, 9, 10]. In [6] Ozturk and Jun have introduced the concept of symmetric bi-derivation of a near ring and studied some properties. In [3] further Ceven and Ozturk introduce the concepts of symmetric bi-(σ,τ)-derivation of a near ring and gave some properties. In [7] Ozturk and Yazarli have studied some results on symmetric bi-(σ,τ)-derivations in near-rings to semi-group ideals of 3-prime near-rings. In [2] Bresar have studied centralizing mappings and derivations in prime rings. In this paper we studied some results on symmetric bi-(σ,τ)- derivations in prime rings.

Preliminaries: Throughout this paper R will be a prime ring and the center of R will be denoted by Z, σ and τ be a automorphisms of R. Recall that a ring R is prime if $xRy = \{0\}$ implies $x = 0$ or $y = 0$. A ring R to be n-torision free if $nx = 0$ implies $x = 0$, for all $x \in R$. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$. A mapping $F(.,.): R \times R \to R$ is called symmetric bi-additive if $F(x, y) = F(y, x)$, for all $x, y \in R$. A mapping $f: R \to R$ is said to be trace of F if $f(x) = F(x, x)$, for all $x \in R$, where $F(.,.,.): R \times R \to R$ is a symmetric bi-additive mapping. The trace of F satisfies the relation $f(x + y) = f(x) + 2F(x, y) + f(y)$, for all $x, y \in R$. An additive mapping $d: R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. A Symmetric bi-additive mapping $D(.,.): R \times R \to R$ is called a symmetric bi-derivation if $D(xw, y) = D(x, y)w + xD(w, y)$, for all $x, y, w \in R$. An additive mapping $d: R \to R$ is called a (σ,τ)-derivation if $d(xy) = d(x)\sigma(y) + \tau(x)d(y)$, for all $x, y \in R$, where σ and τ to be a automorphisms of R. A symmetric bi-additive mapping $D: R \times R \to R$ is called a symmetric bi-(σ,τ)-derivation if there exists functions $\sigma, \tau: R \to R$ such that $D(xw, y) = D(x, y)\sigma(w) + \tau(x)D(w, y)$, for all $x, y, w \in R$.

Throughout this paper, we shall make use of the some basic commutator identities:

$[x, yz] = y[x, z] + [x, y]z$; $[xy, z] = [x, z]y + x[y, z]$; $[x, y]_{\sigma,\tau} = x\sigma(y) - \tau(y)x$.

Remarks:

1. The non zero elements from $Z(R)$ are non zero divisers.
2. If d is non zero derivation of R then d does not vanish on a nonzero left ideal of R.
3. If R contains a commutative non zero left ideal, then R is commutative.
4. Let c and ac be in a center of R. If c is not zero then $a$is in center of R.
5. R has no non zero nil left ideals of bounded index.

6. If $a, b \in R$ such that $axb = bxa$, for all $x \in R$, and if $a \neq 0$, then $b = \mu a$ for some μ in the extended centorid of R.

Lemma1: Let R be a 2-torsion free prime ring. Suppose there exists Symmetric bi-(σ, τ)-derivations $G(.,.): R \times R \to R$ and $H(.,.): R \times R \to R$ such that $g(x)h(y) = h(x)g(y)$, for all $x, y \in R$ where σ, τ are automorphisms on R. If $g \neq 0$ then there exists $\mu \in C$ such that $h(x) = \mu g(x)$, for all $x \in R$.

Proof: We have $g(x)h(y) = h(x)g(y)$, for all $x, y \in R$. (1)

We replaced x by $x + t$ in (1), we get

$g(x+t)h(y) = h(x+t)g(y)$

$\big(g(x) + g(t) + 2G(x,t)\big)h(y) = \big(h(x) + h(t) + 2H(x,t)\big)g(y)$

$g(x)h(y) + g(t)h(y) + 2G(x,t)h(y) = h(x)g(y) + h(t)g(y) + 2H(x,t)g(y)$

$2G(x,t)h(y) = 2H(x,t)g(y)$

$\Rightarrow G(x,t)h(y) = H(x,t)g(y)$, for all $x, y, t \in R$. (2)

We replace t by xt in equation (2), we get

$G(x,xt)h(y) = H(x,xt)g(y)$,

$(G(x,x)\sigma(t) + \tau(x)G(x,t))h(y) = (H(x,x)\sigma(t) + \tau(x)H(x,t))g(y)$

$G(x,x)\sigma(t)h(y) + \tau(x)G(x,t)h(y) = H(x,x)\sigma(t)g(y) + \tau(x)H(x,t)g(y)$

By using (2), we get

$g(x)\sigma(t)h(y) = h(x)\sigma(t)g(y)$, for all $x, y, t \in R$. (3)

$g(x)\sigma(t)h(x) = h(x)\sigma(t)g(x)$, for all $x, y, t \in R$. (4)

If $g(x) \neq 0$, by using remark 6, then exits $\mu(x)$ such that

$h(x) = \mu(x)g(x), \forall\, x \epsilon R$. (5)

If $g(y) \neq 0$ by using remark 6, then exists $\mu(y)$ such that such that

$h(y) = \mu(y)g(y), \forall y \epsilon R$. (6)

We substitute (5) in (3), we get

$g(x)\sigma(t)\mu(y)g(y) = \mu(x)g(x)\sigma(t)g(y)$, for all $x, y, t \in R$

$\Rightarrow \big(\mu(y) - \mu(x)\big)g(x)\sigma(t)g(y) = 0$, for all $t \in R$.

$\big(\mu(y) - \mu(x)\big)g(x)Rg(y) = 0$, for all $t \in R$.

Since R is prime, we get either $\big(\mu(y) - \mu(x)\big)g(x) = 0$, for all $x, y \in R$ or $g(x) = 0$, for all $x \in R$. Now let $A = \{x \in R / \big(\mu(y) - \mu(x)\big)g(x) = 0$, for all $y \in R$ and $B = \{x \in R / g(x) = 0\}$. Clearly, A and

B are additive proper subgroups of R whose union is R. Since a group cannot be the set theoretic union of two proper subgroups. Hence either $A = R$ or $B = R$.
If $A = R$, then $\big(\mu(y) - \mu(x)\big)g(x) = 0$, for all $x, y \in R$.

Since $g(x) \neq 0$, it follows that, $\mu(y) - \mu(x) = 0$, for all $x, y \in R$.

Which implies that $\mu(y) = \mu(x)$, for all $x, y \in R$.

From equation (5) and equation (6), we get

$h(x) = \mu g(x), \forall\ x \epsilon R$.

On the other hand if $B = R$, then $g(x) = 0$, for all $x \in R$, then from equation (1), we get $h(x) = 0$, for all $x \in R$. Hence there exists $\mu \in C$ such that $h(x) = \mu g(x)$, for all $x \in R$.

Lemma2: Let R be a 2-torsion free prime ring. Suppose there exists symmetric bi-(σ, τ)-derivations $F(.,.): R \times R \to R$, $G(.,.): R \times R \to R$, $H(.,.): R \times R \to R$ and $T(.,.): R \times R \to R$ such that $f(x)h(y) = t(x)g(y)$, for all $x, y \in R$ and f, g, h, t are traces of F, G, H, T respectively. If $g \neq 0$ and $f \neq 0$ then there exists $\mu \in C$ such that $h(x) = \mu g(x)$ and $t(x) = \mu f(x)$, for all $x \in R$.

Proof: we have $f(x)h(y) = t(x)g(y)$, for all $x, y \in R$. (7)

We replace x by $x + z$ in (7), we get

$f(x+z)h(y) = t(x+z)g(y)$

$\big(f(x) + f(z) + 2F(x,z)\big)h(y) = (t(x) + t(z) + 2T(x,z))g(y)$

$\big(f(x)h(y) + f(z)h(y) + 2F(x,z)h(y)\big) = t(x)g(y) + t(z)g(y) + 2T(x,z)g(y)$

Using (7), we get

$F(x,z)h(y) = T(x,z)g(y)$, for all $x, z \in R$. (8)

We replace z by xz in (8), we get

$F(x,xz)h(y) = T(x,xz)g(y)$

$F(x,x)\sigma(z)h(y) + \tau(x)F(x,z)h(y) = T(x,x)\sigma(z)g(y) + \tau(x)T(x,z)g(y)$

Using (8), we get

$f(x)\sigma(z)h(y) = t(x)\sigma(z)g(y)$), for all $x, y, z \in R$. (9)

We replace $\sigma(z)$ by $\sigma(z)g(w)$ in (9),we get

$f(x)\sigma(z)g(w)h(y) = t(x)\sigma(z)g(w)g(y)$. (10)

We replace y by w in (9), we get

$f(x)\sigma(z)h(w) = t(x)\sigma(z)g(w)$, for all $x, y, z \in R$. (11)

From (11) & (10) we have

$f(x)\sigma(z)g(w)h(y) = f(x)\sigma(z)h(w)g(y)$

$\Rightarrow f(x)\sigma(z)\big(g(w)h(y) - h(w)g(y)\big) = 0$

Since $f \neq 0$ and R is prime which implies

$g(w)h(y) - h(w)g(y) = 0$

$\Rightarrow g(w)h(y) = h(w)g(y)$

We assume that $g \neq 0$ and hence it follows from lemma 1, we get $h(y) = \mu g(y)$, for all $x, y, z\epsilon R$ since $f \neq 0$ and R is prime, which implies that $\mu f(x) - t(x) = 0$. $t(x) = \mu f(x)$, for all $x \in R$.

Theorem1: Let R be a 2-torsion free prime ring. Suppose there exists a symmetric (σ,τ) bi-derivations $G(.,.): R \times R \to R$, $H(.,.): R \times R \to R$, and $T(.,.): R \times R \to R$ and $a, b \in R$ such that $g(x) = ah(x) + t(x)b$, for all $x \in R$. If $a \notin Z(R)$ and $b \notin Z(R)$ then there exists $\mu \in C$ such that $g(x) = [2\mu ab, a\sigma(x) + \tau(x)b], h(x) = [\mu b, \sigma(x)]$ and $t(x) = [\mu(a), \tau(x)]$, for all $x \in R$, where g, h and t are trace of G, H and T respectively.

Proof: we have $g(x) = ah(x) + t(x)b$, for all $x \in R$. (12)

We replace x by $x + y$ in equation (12), we get

$g(x+y) = ah(x+y) + t(x+y)b$

$g(x) + g(y) + 2G(x,y) = a\big(h(x) + h(y) + 2H(x,y)\big) + \big(t(x) + t(y) + 2T(x,y)\big)b$ By using (12) in the above equation, we get

$2G(x,y) = 2aH(x,y) + 2T(x,y)b$

$G(x,y) = aH(x,y) + T(x,y)b$, for all $x, y \in R$. (13)

We replace x by xy in equation (13), we get

$G(xy,y) = aH(xy,y) + T(xy,y)b$

$G(x,y)\sigma(y) + \tau(x)G(y,y) = aH(x,y)\sigma(y) + a\tau(x)H(y,y) + T(x,y)\sigma(y)b + \tau(x)T(y,y)b$

$G(x,y)\sigma(y) + \tau(x)g(y) = aH(x,y)\sigma(y) + a\tau(x)h(y) + T(x,y)\sigma(y)b + \tau(x)t(y)b$

By using (13) in the above equation we get

$\tau(x)g(y) = a\tau(x)h(y) + \tau(x)t(y)b$, for all $x, y \in R$. (14)

Multiply (12) by $\tau(x)$ on left hand side, we get

$\tau(x)g(y) = \tau(x)ah(y) + \tau(x)t(y)b$, for all $x, y \in R$. (15)

Subtract (15) from (14), we get

$[a, \tau(x)]h(y) = 0$, for all $x, y \in R$. (16)

We replace y by xy in (13), we get

$G(x,xy) = aH(x,xy) + T(x,xy)b$

$G(x,x)\sigma(y)+\tau(x)G(x,y)=a\big(H(x,x)\sigma(y)+\tau(x)H(x,y)\big)+\big(T(x,x)\sigma(y)+\tau(x)T(x,y)\big)b$

$g(x)\sigma(y)+\tau(x)G(x,y)=ah(x)\sigma(y)+a\tau(x)H(x,y)+t(x)\sigma(y)b+\tau(x)T(x,y)b$

By using (13) in the above equation, we get

$g(x)\sigma(y)=ah(x)\sigma(y)+t(x)\sigma(y)b+\tau(x)T(x,y)b$, for all $x,y\in R$. (17)

Multiplying (12) by $\sigma(y)$ on right side, we get

$g(x)\sigma(y)=ah(x)\sigma(y)+t(x)b\sigma(y)$, for all $x,y\in R$. (18)

We subtract (17) from (18), we get

$t(x)(b\sigma(y)-\sigma(y)b)=0$

$t(x)[b,\sigma(y)]=0$, for all $x,y\in R$. (19)

We subtract (19) from (16), we get

$[a,\tau(x)]h(y)-t(x)[b,\sigma(y)]=0$

$[a,\tau(x)]h(y)=t(x)[b,\sigma(y)]$, for all $x,y\in R$

Now by lemma 2 there exists $\mu\epsilon C$ such that $t(x)=[\mu a,\tau(x)]$ and $h(x)=[\mu b,\sigma(x)]$, for all $x\in R$

Hence by (12) yields that $g(x)=a[\mu b,\sigma(x)]+[\mu a,\tau(x)]b$

$g(x)=[2\mu ab, a\sigma(x)+\tau(x)b]$, for all $x\in R$.

Theorem2: Let R be 2-torsion free prime ring. Suppose there exists symmetric (σ,τ) bi-derivation $G(.,.):R\times R\rightarrow R$ and $H(.,.):R\times R\rightarrow R$ and $a,b\in R$ such that$ag(x)+h(x)b=0$, for all $x\in R$. If $a\notin Z(R)$ and $b\notin Z(R)$ then there exists $\mu\epsilon C$ such that $g(x)=[\mu b,x]$ and $h(x)=[\mu a,x]$ for all $x\epsilon R$. More over if $g\neq 0$ implies $ab\epsilon Z(R)$.

Proof: we have $ag(x)+h(x)b=0$, for all $x\in R$. (20)

We replace x by $x+y$ in equation (20), we get

$ag(x+y)+h(x+y)b=0$

$a\big(g(x)+g(y)+2G(x,y)\big)+\big(h(x)+h(y)+2H(x,y)\big)b=0$

$2aG(x,y)+2H(x,y)b=0$

$aG(x,y)+H(x,y)b=0$, for all $x,y\in R$. (21)

We replace x by xy in (21), we get

$aG(xy,y)+H(xy,y)b=0$

$a\big(G(x,y)\sigma(y)+\tau(x)G(y,y)\big)+\big(H(x,y)\sigma(y)+\tau(x)H(y,y)\big)b=0$

Using (21), we get

$a\tau(x)g(y) + \tau(x)h(y)b = 0$, for all $x, y \epsilon R$. (22)

Multiply (20) by $\tau(x)$ on left hand side, we get

$\tau(x)ag(y) + \tau(x)h(y)b = 0$, for all $x, y \epsilon R$. (23)

We subtract (23) from (22), we get

$a\tau(x)g(y) - \tau(x)ag(y) = 0$

$[a, \tau(x)]g(x) = 0$, for all $x \epsilon R$. (24)

We replace y by xy in (21), we get

$aG(x, xy) + H(x, xy)b = 0$

$aG(x, x)\sigma(y) + a\tau(x)G(x, y) + H(x, x)\sigma(y)b + \tau(x)H(x, y)b = 0$

$ag(x)\sigma(y) + h(x)\sigma(y)b = 0$, for all $x, y \in R$. (25)

Multiply (20) by $\sigma(y)$ on right hand side, we get

$ag(x)\ \sigma(y) + h(x)b\ \sigma(y) = 0$, for all $x, y \in R$. (26)

We subtract (25) for (26), we get

$h(x)b\sigma(y) - h(x)\sigma(y)b = 0$

$h(x)(b\sigma(y) - \sigma(y)b) = 0$

$h(x)[b, \sigma(y)] = 0$, for all $x, y \in R$. (27)

We subtract (27) from (24), we get

$[a, \tau(x)]g(x) = h(x)[b, \sigma(y)]$

Now by lemma 2, there exist $\mu \epsilon C$ such that $h(x) = [\mu a, \tau(x)]$ and $g(x) = [\mu b, \sigma(x)]$, for all $x \in R$.

If $g \neq 0$ then $\mu \neq 0$ and so $ag(x) + h(x)b = 0$

$a[\mu b, \sigma(x)] + [\mu a, \sigma(x)]b = 0$

$[2\mu ab, a\sigma(x) + \tau(x)b] = 0$

$\mu ab \epsilon Z(R)$ then $ab \epsilon Z(R)$.

Theorem3: let R be non commutative 2-torsion free prime ring and U be a non zero left ideal of R. Suppose there exists symmetric (σ, τ) bi –derivation $G(.,.): R \times R \to R$ and g is a trace of G. If g of R maps U in the center $Z(R)$, then $g = 0$.

Proof: Let $x, y \epsilon U$, then $x + y \in U$

Which implies that $g(x), g(y), g(x + y) \in Z(R)$, for all $x, y \epsilon R$

$[g(x), x] = 0$

and $[g(x+y),x]=0$, for all $x,y\in U$. (28)

$[g(x)+g(y)+2G(x,y),x]=0$

$[g(x),x]+[g(y),x]+2[G(x,y),x]=0$

$[g(y),x]+2[G(x,y),x]=0$, for all $x,y\epsilon R$. (29)

x by $-x$ in (29), we get

$-[g(y),x]+2[G(x,y),x]=0$, for all $x,y\epsilon R$. (30)

By adding (29) and (30), we get

$4[G(x,y),x]=0$

$[G(x,y),x]=0$, for all $x,y\epsilon R$ (31)

Substitute y by xy in (31), we get

$[G(x,xy),x]=0$

$[G(x,x)\sigma(y)+\tau(x)G(x,y),x]=0$

$[g(x)\sigma(y),x]+[\tau(x)G(x,y),x]=0$

$g(x)[\sigma(y),x]+[g(x),x]\sigma(y)+\tau(x)[G(x,y),x]+[\tau(x),x]G(x,y)=0$, for all $x,y\epsilon U$

By using (28) and (31) in the above equation, we get

$g(x)[\sigma(y),x]=0$

From remark 1 it follows $g(x)=0$ or $x\in Z(U)$.
In other word U is a union of its subsets, $G=\left\{\frac{x\epsilon U}{g(x)}=0\right\}$ and $H=\{x\epsilon U/[\sigma(y),x]=0\}$ note that both are additive subgroups of U. But the group cannot be the union of two proper subgroup then either $G=H$ or $H=U$. If $H=U$ then U is commutative which is impossible by remark3. Hence $G=U$ and using remark 2 we obtain of the theorem.

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